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N-Person Game Theory: Concepts and Applications, by Anatol Rapoport

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BOOK REVIEW

N-PERSON GAME THEORY: CONCEPTS AND APPLICATIONS. By Anatol
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*The good Christian should be wary of mathematicians and
all those who make a practice of sacrilegious predictions,
particularly when they speak the truth. Because the danger
exists that these people, in league with the devil, may becloud
the souls of men and enmesh them in the snares of hell.*[1]

I.

A game is a set of rules describing possible actions by individuals
or groups and the consequences of such actions. The rules normally
designate a set of players, define the behavioral choices open to the
players and the information on which these choices must be based, and
associate with each combination of choices an outcome comprising a
payoff to each player. Game theory, the study of games, was established a
quarter of a century ago as a branch of mathematics thought to have
particularly obvious economic applications;[2] recent work has suggested
its value in such disparate disciplines as anthropology[3] and the philosophy
of language.[4] Discontinuities in terms of applicable mathematical tech-
niques led in the initial stages of interest in game theory to isolation of
games involving two players as the focus of particular scholarly attention.
Rapoport, senior research mathematician and professor of mathematical
biology in the Mental Health Research Institute at the University of
Michigan, has previously provided an introductory survey of resulting
contributions.[5] More recently the interest of both mathematicians and
social scientists has been increasingly drawn toward generalizations in-
volving three or more players. *N-Person Game Theory* evidences this
shift but also supplies cause for its acceleration: the author makes the
products of specialized research available to a wide audience by supple-
menting his substantive overview with a thirty-page introduction to the
mathematics he employs and other generous methodological assistance.

1. AUGUSTINE, DE GENESI AD LITTERAM 2, XVII, 37, *quoted in* Weissinger, *The
Characteristic Features of Mathematical Thought*, THE SPIRIT AND THE USES OF THE
MATHEMATICAL SCIENCES 9 (T. Saaty & F. Weyl eds. 1969).
    2. J. VON NEUMANN & O. MORGENSTERN, THEORY OF GAMES AND ECONOMIC BE-
HAVIOR (rev. ed. 1953). *See generally* R. LUCE & H. RAIFFA, GAMES AND DECISIONS:
INTRODUCTION AND CRITICAL SURVEY (1957).
    3. GAME THEORY IN THE BEHAVIORAL SCIENCES (I. Buchler & H. Nutini eds. 1969).
    4. D. LEWIS, CONVENTION: A PHILOSOPHICAL STUDY (1969).
    5. A. RAPOPORT, TWO-PERSON GAME THEORY: THE ESSENTIAL IDEAS (1966).

The relevance of game theory to the law will remain untested so long as the language in which game theory is expressed is unknown to the lawyer. To posit its value is to endorse a study which must be justified primarily by appeal to rewards of increased understanding which cannot be adequately demonstrated by description. The proponent is placed in a position reminiscent of that of Galileo, whose efforts to demonstrate the existence of Jupiter's moons by persuading doubters to view them through his telescope proved largely abortive, since many either refused to look on grounds of religious principle or denounced what they saw as an illusion inspired by the devil. In the next section of this review I indicate what must be learned before relevance can be determined by outlining aspects of Rapoport's work. I subsequently support a claim of the potential importance of this work to the law by suggesting its applicability in a fundamental area of legal analysis. In structure my approach parallels that of Rapoport, who in the body of the book discusses separately basic concepts and their uses. My presentation is in both sections necessarily incomplete: the first conveys at most the flavor of the discipline; the second proposes rather than proves its utility.

II.

Rapoport states:

We now define the N-person game as follows. A certain amount is to be divided among the players. Next, it is specified how much each subset of players could get if they simply declared themselves to be a coalition, regardless of what others get or how they are organized. The problem to be solved is what to expect in the way of coalition formation and apportionment of payoffs.[6]

A coalition is a group of one or more players the members of which, if there are more than one, cooperate to maximize their joint payoff. If the players are identified by number, one may designate a coalition of for instance the first and second players by $\overline{12}$. The function $v(\ )$, the characteristic function of a game, specifies the value of that game in transferable units such as money to each of all possible coalitions of players. Payoffs to the players individually may be written as a vector:

$$\vec{x} \equiv (x_1, x_2, x_3, \ldots, x_n).$$

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6. A. RAPOPORT, N-PERSON GAME THEORY: CONCEPTS AND APPLICATIONS 137 (1970) (hereinafter RAPOPORT).

N-person game theory examines the relationship between the characteristic function of a game and various combinations of payoffs and coalitions.

The parameters of analysis are best adduced by example. Let a three-person game yield no advantage to solitary action but reward cooperation between any two players or among the three players with a joint gain of one unit. One may write:

$$v(\bar{1}) = v(\bar{2}) = v(\bar{3}) = 0$$
$$v(\overline{12}) = v(\overline{13}) = v(\overline{23}) = 1$$
$$v(\overline{123}) = 1.$$

In this section I discuss in the context of this game three controlling concepts stressed by Rapoport, those of imputation, core, and Shapley value.

A payoff vector is an imputation if it offers no player less than he could obtain by acting alone and if the amounts offered to the players sum to the value of the associated game to a coalition of all the players. An imputation in a game with N players thus satisfies these conditions:

$$x_i \geqslant v(\bar{i}). \quad (i = 1, 2, \ldots, n)$$

$$\sum_{i=1}^{n} x_i = v(N).$$

In the game of the preceding paragraph any nonnegative vector distributing a payoff of one unit among the players is an imputation. Here the set of imputations may be represented diagramatically as the points within or along the perimeter of a triangle the corners of which may be taken to be the limiting cases giving all gain to one player:

(0, 1, 0)

(1, 0, 0)   (0, 0, 1)
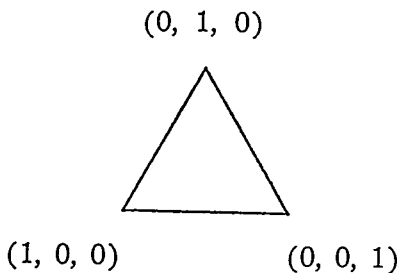
The core of a game consists of those imputations which are stable in the sense that no group of players can through isolated action increase the payoffs to its members. Let S be an arbitrary subset of N. Then an imputation is in the core if for all S:

$$v(S) \leqslant \sum_{i \in S} x_i$$

As Rapoport notes, "[t]he core solution says that the amount $v(N)$ will be apportioned in such a way that no subset will be motivated to leave the grand coalition by what the characteristic function awards to them as a coalition of their own."[7] The illustrative game has an empty core, since irrespective of the division of the payoff two players must together receive less than one unit and a combination of these players can demand this amount. If the value of the game to coalitions of two players were two thirds rather than one, the single imputation (1/3, 1/3, 1/3), the midpoint of the triangle, would constitute the core; if the value to such coalitions were zero, all imputations would be in the core. The concept of the core only coincidentally isolates a unique distributional result: more typically the core either is empty or contains more than a single imputation.

The expected contribution of a player to a randomly forming coalition is termed the Shapley value to him of the game in which he is participating. In a three-person game the players can unite in any of six orders, assumed equally probable: (1, 2, 3), (1, 3, 2), (2, 1, 3), (3, 1, 2), (2, 3, 1), or (3, 2, 1). The contribution of a player is in each case the difference between the values of the game to the coalition after and before his participation. In the illustrative game the second player to enter a coalition increases joint gain by one unit, while the first and third add nothing. Since the value of the game to a coalition does not depend on the identity of those cooperating but only on their number, the Shapley values of the players are undifferentiated. Player one, for example, offers nothing to coalitions formed in orders (1, 2, 3), (1, 3, 2), (2, 3, 1), and (3, 2, 1) but contributes one unit to coalitions formed in orders (2, 1, 3) and (3, 1, 2). Player two supplies benefit only when coalitions are formed in order (1, 2, 3) or order (3, 2, 1); player three adds nothing unless coalitions are formed in order (1, 3, 2) or order (2, 3, 1). Be-

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7. *Id.* at 138.

cause each player contributes one unit in two of six situations the Shapley value of the game to each player is one third. If the characteristic function of the game were to associate no advantage rather than a gain of one unit with a coalition of players two and three, the augmented contribution of player one would be reflected in the Shapley values by assignment to him of a larger proportion of the joint benefit. Here he would add one unit not only to coalitions which he is the second player to join but also to coalitions which he is the third player to join; since he would now supply a gain of one unit in four of six situations, the Shapley value to him of the transformed game is two thirds. Players two and three would share equally what remains: because they would add one unit to coalitions formed in orders (1, 2, 3) and (1, 3, 2) respectively the Shapley value of the transformed game to each of them is one sixth. The sum of the Shapley values to all of the players of any game exactly exhausts the gain available to a coalition including every player. The solution is in every instance unambiguous.

III.

Rapoport applies the theory he presents primarily to economic and political problems: as he suggests, the characteristic function supplies an appealing framework for examination of the effects of competition and monopoly in market structures; in addition, its coalitional premises make it an ideal vehicle for analysis of voting behavior. Although both of these uses are at least peripherally relevant to the law, pertinence is more clearly demonstrated in other contexts. For example, reinterpretation of the concept of contract as a response to inefficiencies generated by the game described in the preceding section and its variants can provide insights into both the worth and the limitations of this legal instrument.

Discussing games such as that above, Rapoport asserts:

> When the core is empty, it means that whatever coalition structure obtains, there is always some subset of players, not all in the same coalition, who would rather be in a coalition, since they can get more that way than with the existing structure. If no restrictions are put on the way coalitions are formed, the disgruntled subset of players *can* form a coalition and so disrupt the old structure. But in the new structure there is certain to be another subset of players, not all in the same coalition, who would rather be in a coalition for the same reason as before. In other words, no coalition structure is stable if the core is empty.[8]

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8. *Id.*

The law of contract facilitates stabilization of otherwise coreless games by restricting inducement to the players to restructure their allegiances. Thus in the illustrative game the integrity of any coalition of two or three players dividing a gain of one unit can be assured by the rule that "where a party sustains a loss by reason of breach of contract, he is, so far as money can do it, to be placed in the same situation, with respect to damages, as if the contract had been performed."[9] This rule, guaranteeing to an excluded player his initial payoff, discourages realignment by precluding resulting gain. One may "view the law of contract as directed to strengthening the security of transactions by enabling men to rely more fully on promises . . . ."[10]

The general damage remedy for breach of contract, which seeks to place the injured party in as good a position as he would have occupied if the agreement had been performed, has usually been defended through appeal to subjective concepts of natural justice. A less introspective justification is available: this measure is also dictated by considerations of economic efficiency, since it encourages optimal reallocation of factors of production and goods without causing material instability of expectations. Consider a game which differs from the illustrative game only in that its values to coalitions $\overline{12}$ and $\overline{13}$ are reduced to one third and two thirds respectively:

$$v(\overline{1}) = v(\overline{2}) = v(\overline{3}) = 0$$
$$v(\overline{12}) = 1/3; \, v(\overline{13}) = 2/3; \, v(\overline{23}) = 1$$
$$v(\overline{123}) = 1.$$

Here coalition $\overline{12}$ is unstable irrespective of contractual support, since either member and the excluded player can combine to obtain a joint payoff more than sufficient both to pay damages to the deserted player and to maintain the level of gain of the withdrawing player. Coalition $\overline{13}$ is similarly unstable, although only one of its members is motivated to withdraw. In both cases formation of the grand coalition can benefit all players. The law generally sacrifices stability to efficiency when the two values conflict, directing that profit made through breach of contract above that necessary to satisfy compensatory damage claims be retained by the breaching party. But it dictates that "a rearrangement of rights will only be undertaken when the increase in the value of production consequent upon the rearrangement is greater than the costs which would

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9. Robinson v. Harmon, 154 Eng. Rep. 363 (Ex. 1848).
10. M. COHEN, LAW AND THE SOCIAL ORDER 102 (1933).

be involved in bringing it about."[11]

The law of contract is nevertheless unsatisfactory to the extent that it invites rationalization of the distributive inequities possible within its framework. Its constraints in the illustrative game do not preclude but indeed protect imputations such as (1/2, 1/2, 0) which deprive an excluded player of all benefit. The disparity between this result and the Shapley value, often proferred as the just solution, is particularly striking in this context. Social parallels are evidenced by critics who admit that "the contract law system may be seen as helping to maintain the smooth working of the economic system"[12] but protest:

> If an economic system may be said to have failed when millions of people are in poverty, though goods are abundant, then the smooth maintenance of our system is the smooth maintenance of a failed system. Take this at its least: Insofar as it contributes to lubricating our economic system, considered as a set of bargains, the contract law system is contributing nothing whatever to the solution of our most pressing problem of economic justice. But the plot thickens. For the contract law system . . . serves massively and systematically as an *intensifier* of economic advantage and disadvantage. It does this because people and businesses who are in strong bargaining positions, or who can afford expensive legal advice, can and epidemically do exact of necessitous and ignorant people contractual engagements which the general law never would impose. . . . The best one could say of it is that no amount of restudy of doctrine, no overhaul in the light of social fact, could ever make this system *affirmatively* responsive to the needs which now cry out so loudly for relief that other social demands ought hardly to be heard.[18]

IV.

The potential value of game theory to the law inheres in its abstractness; it can facilitate study of the consequences of legal rules by providing a perspective undistorted by accidental complexities. This service would parallel that already supplied to other disciplines. Rapoport, who interprets game theory as an inquiry into the logic of conflict situations, contends: "This logic turns out to be intricate and often perplexing, at times ridden with paradoxes, which, when resolved, provide us with

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11. Coase, *The Problems of Social Cost*, 3 J. LAW & ECON. 1, 15-16 (1960).
12. Black, *Some Notes on Law Schools in the Present Day*, 79 YALE L.J. 505, 507 (1970).
13. *Id.* at 507-09.

insight concerning matters which had been either ignored or only vaguely understood."[14] If, as he maintains, "[t]he primary function of such analysis is . . . to clarify and illuminate the issues in a decision problem,"[15] it would seem well suited to use by lawyers and legal scholars. His book provides an excellent introduction to the requisite methodology.

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14. RAPOPORT 52.
15. *Id.* at 124.
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